

EXACT VALUES OF KOLMOGOROV WIDTHS OF CLASSES OF POISSON INTEGRALS

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Abstract

We prove that the Poisson kernel $P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos\left(kt - \frac{\beta\pi}{2}\right)$, $q \in (0, 1)$, $\beta \in \mathbb{R}$ satisfies introduced by Kyshpel' condition $C_{y,2n}$ beginning from some number n_q that depends only on q . As a consequence, the lower bounds for Kolmogorov widths in the space C of classes $C_{\beta,\infty}^q$ of Poisson integrals of functions from unit ball in space L_∞ are found for all $n \geq n_q$. These estimates coincide with the best uniform approximation of mentioned classes by trigonometric polynomials. As a result, it is found the exact values of the widths of classes $C_{\beta,\infty}^q$ and shown that the subspaces of trigonometric polynomials of order $n - 1$ are optimal for the widths of dimension $2n - 1$.

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Let $L = L_1$ be the space of 2π -periodic Lebesgue summable functions f with the norm $\|f\|_1 = \int_{-\pi}^{\pi} |f(t)|dt$, let L_∞ be the space of 2π -periodic functions, which are Lebesgue measurable and essentially bounded, with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{t \in \mathbb{R}} |f(t)|$, and let C be the space of 2π -periodic continuous functions f , where the norm is defined by the formula $\|f\|_C = \max_{t \in \mathbb{R}} |f(t)|$.

Further, let $\Psi_\beta(t)$ be a fixed summable kernel of the form

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \psi(k) > 0, \quad \sum_{k=1}^{\infty} \psi(k) < \infty, \quad \beta \in \mathbb{R}. \quad (1)$$

We denote by $C_{\beta,p}^\psi$, $p = 1, \infty$, the class of 2π -periodic functions f , which can be

represented as the convolution with kernel Ψ_β

$$f(x) = A + \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi_\beta(x-t) \varphi(t) dt = A + (\Psi_\beta * \varphi)(x), \quad A \in \mathbb{R}, \quad (2)$$

where

$$\|\varphi\|_p \leq 1, \quad \varphi \perp 1.$$

The function φ in relation (2) is called the (ψ, β) -derivative of the function f and denoted by f_β^ψ . The notion of (ψ, β) -derivative was introduced by A.I. Stepanets (see, for example, [1, chapter 3, sections 7–8]).

The important partial case of kernels $\Psi_\beta(t)$ of the form (1) for $\psi(k) = q^k$, $q \in (0, 1)$, is Poisson kernels $P_{q,\beta}(t)$ with parameters q and β , i.e., functions

$$P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos \left(kt - \frac{\beta\pi}{2} \right), \quad q \in (0, 1), \quad \beta \in \mathbb{R}.$$

Functions f , which can be represented as the convolution (2) with kernel $\Psi_\beta(t) = P_{q,\beta}(t)$, are called the Poisson integrals. In this case we will denote the classes $C_{\beta,p}^\psi$ by $C_{\beta,p}^q$ and (ψ, β) -derivative f_β^ψ of function $f \in C_{\beta,p}^\psi$ for $\psi(k) = q^k$ — by f_β^q .

Denote by $E_n(C_{\beta,p}^\psi)_X$, where $p = 1, \infty$ and $X = L$ or C respectively, the best approximations in space X for the class $C_{\beta,p}^\psi$ by subspace \mathcal{T}_{2n-1} of trigonometric polynomials t_{n-1} of order $n-1$, i.e., quantity of the form

$$E_n(C_{\beta,p}^\psi)_X = \sup_{f \in C_{\beta,p}^\psi} \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f - t_{n-1}\|_X, \quad (3)$$

and by $d_m(C_{\beta,p}^\psi, X)$ we denote Kolmogorov width of order m for class $C_{\beta,p}^\psi$ in the space X , i.e., quantity of the form

$$d_m(C_{\beta,p}^\psi, X) = \inf_{F_m \subset X} \sup_{f \in C_{\beta,p}^\psi} \inf_{y \in F_m} \|f - y\|_X, \quad (4)$$

where the outer infimum is taken over all linear subspaces $F_m \subset X$ of dimension m .

The problem of obtaining Kolmogorov widths for various functional compacts in various functional spaces has a rich history, which can be found in monographs [2, 3].

In the present paper we consider the problem of obtaining exact values of widths $d_{2n}(C_{\beta,\infty}^q, C)$, $d_{2n-1}(C_{\beta,\infty}^q, C)$ and $d_{2n-1}(C_{\beta,1}^q, L_1)$ for all natural n greater some number, which depends only on q .

Note that the problem of obtaining exact values for the widths of classes of Poisson integrals $C_{\beta,p}^q$, $p = 1, \infty$, was considered in works [4–7]. However, in the given subjects a number of unsolved issues remains. The answer for a significant part of it has been obtained in this paper.

As for the widths of sets of Poisson integrals of functions from classes H_ω generated by modulus of continuity, the exact or asymptotically exact estimates were obtained only in some cases (see, for example, [8, 9]).

For quantities of kind (3) and (4) next relationship holds:

$$d_{2n-1}(C_{\beta,p}^\psi, X) \leq E_n(C_{\beta,p}^\psi)_X. \quad (5)$$

As follows from [6, 10–12], for arbitrary $q \in (0, 1)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\begin{aligned} E_n(C_{\beta,\infty}^q)_C &= E_n(C_{\beta,1}^q)_{L_1} = \|P_{q,\beta} * \varphi_n\|_C = \\ &= \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} \sin \left((2\nu+1)\theta_n\pi - \frac{\beta\pi}{2} \right) \right|, \end{aligned} \quad (6)$$

where

$$\varphi_n(t) = \operatorname{sgn} \sin nt, \quad (7)$$

and $\theta_n = \theta_n(q, \beta)$ is the unique on $[0, 1)$ root of the equation

$$\sum_{\nu=0}^{\infty} q^{(2\nu+1)n} \cos \left((2\nu+1)\theta_n\pi - \frac{\beta\pi}{2} \right) = 0. \quad (8)$$

Therefore, to solve the problem of obtaining the exact values of mentioned widths it remains to establish lower bounds of

$$d_{2n}(C_{\beta,\infty}^q, C) \geq \|P_{q,\beta} * \varphi_n\|_C, \quad (9)$$

$$d_{2n-1}(C_{\beta,1}^q, L) \geq \|P_{q,\beta} * \varphi_n\|_C. \quad (10)$$

Obtaining of estimates (9) and (10) is associated with the principal difficulties, caused by the fact that the Poisson kernel $P_{q,\beta}(t)$ may increase the oscillations (e.g., as shown in [4, p. 1318–1319], $P_{q,\beta} \notin \text{CVD}$ when $q = 1/7$ and $\beta = 0$). Therefore, for the class of convolution with kernels $P_{q,\beta}(t)$ it is impossible to obtain exact lower bounds for widths using methods and approaches developed by A. Pinkus [2].

So far estimates (9) and (10) have been known in the following cases:

- for arbitrary $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$ in case $0 < q \leq q(\beta)$, where $q(\beta) = 0,2$ for $\beta \in \mathbb{Z}$ and $q(\beta) = 0,196881$ for $\beta \in \mathbb{R} \setminus \mathbb{Z}$ (see [6]);
- for $\beta = 2kl$, $l \in \mathbb{Z}$, arbitrary $0 < q < 1$ and all number n greater some number n_* which depends on q (in this case it is proved the existence of n_* but doesn't specify a constructive method to find it) [7].

In these cases, the results were obtained on the basis of using developed by A.K. Kushpel' [4] method of obtaining lower bounds for widths of classes of convolutions with forming kernels Ψ_β satisfying the so-called $C_{y,2n}$ conditions. In this paper, we also hold this approach. Let formulate definitions and known propositions which will be used to present the results of paper.

Let $\Delta_{2n} = \{0 = x_0 < x_1 < \dots < x_{2n} = 2\pi\}$, $x_k = k\pi/n$, be a partition of the segment $[0, 2\pi]$. Consider a function

$$\Psi_{\beta,1}(t) = (\Psi_\beta * B_1)(t) = \sum_{k=1}^{\infty} \frac{\psi(k)}{k} \cos\left(kt - \frac{(\beta+1)\pi}{2}\right), \quad (11)$$

where $B_1 = \sum_{k=1}^{\infty} k^{-1} \sin kt$ is the Bernoulli kernel. Denote by $S\Psi_{\beta,1}(\Delta_{2n})$ the space of SK -splines $S\Psi_{\beta,1}(\cdot)$ with respect to the partition Δ_{2n} , i.e., the set of functions of the form

$$S\Psi_{\beta,1}(\cdot) = \alpha_0 + \sum_{k=1}^{2n} \alpha_k \Psi_{\beta,1}(\cdot - x_k), \quad \sum_{k=1}^{2n} \alpha_k = 0, \quad (12)$$

$$\alpha_k \in \mathbb{R}, \quad k = 0, 1, \dots, 2n.$$

A function $\overline{S\Psi}_{\beta,1}(\cdot) = \overline{S\Psi}_{\beta,1}(y, \cdot)$ of the form (12), that satisfies the conditions

$$\overline{S\Psi}_{\beta,1}(y, y_k) = \delta_{0,k} = \begin{cases} 0, & k = \overline{1, 2n-1}, \\ 1, & k = 0, \end{cases}$$

(here, $y_k = x_k + y$, $x_k = k\pi/n$, $y \in [0, \frac{\pi}{n})$) is called a fundamental SK -spline. Spline $\overline{S\Psi}_{\beta,1}(y, \cdot)$ generates a system of fundamental splines of the form $\overline{S\Psi}_{\beta,1}(y, \cdot - x_k)$, $k = \overline{0, 2n-1}$, that form a basis of the space $S\Psi_{\beta,1}(\Delta_{2n})$. Necessary and sufficient conditions for the existence and uniqueness of the fundamental spline $\overline{S\Psi}_{\beta,1}(y, \cdot)$, depending on the correlation between y (a parameter that determines the shift of interpolation points) and the parameters ψ and β of forming kernel $\Psi_{\beta,1}$, were studied in [4, 13–17].

Since, by virtue of the definition (ψ, β) -derivative, for kernel $\Psi_{\beta,1}$ the equality

$$(\Psi_{\beta,1}(\cdot))_{\beta}^{\psi} = B_1(\cdot) \quad (13)$$

holds and then, as a result of (12), we have

$$(S\Psi_{\beta,1}(\cdot))_{\beta}^{\psi} = \sum_{k=1}^{2n} \alpha_k B_1(\cdot - x_k). \quad (14)$$

Equalities in (13) and (14) are regarded as equalities between two functions from L_1 (i.e., almost everywhere). By virtue of Lemma 2.3.4 of [3, p. 76], the function on the right-hand side of equality (14) is constant on any interval (x_k, x_{k+1}) . Thus, among (ψ, β) -derivatives of any spline of the form (12) and, hence, of the fundamental spline $\overline{S\Psi}_{\beta,1}(\cdot)$, there exists a function constant on any interval (x_k, x_{k+1}) . In what follows, the expression $(\overline{S\Psi}_{\beta,1}(\cdot))_{\beta}^{\psi}$ is regarded as exactly this function.

Definition. We say that for some real number y and the partition Δ_{2n} a kernel $\Psi_{\beta}(\cdot)$ of the form (1) satisfies the $C_{y,2n}$ condition (and write $\Psi_{\beta} \in C_{y,2n}$) if there exists a unique fundamental spline $\overline{S\Psi}_{\beta,1}(y, \cdot)$ for this kernel and the following equality holds:

$$\text{sgn}(\overline{S\Psi}_{\beta,1}(y, t_k))_{\beta}^{\psi} = (-1)^k \varepsilon e_k, \quad k = \overline{0, 2n-1},$$

where $t_k = (x_k + x_{k+1})/2$, the quantity e_k is equal to either 0 or 1, and ε takes the values ± 1 and does not depend on k .

The following theorem allows us to obtain lower bounds for Kolmogorov widths of convolution of classes generated by kernels which satisfy the $C_{y,2n}$ condition.

Theorem 1 (A.K. Kushpel' [4, 5]). *Let Ψ_β be a function of the form (1), that generates classes $C_{\beta,p}^\psi$, $p = 1, \infty$, satisfies the $C_{y,2n}$ condition for some $n \in \mathbb{N}$ where y is the maximum point of the function $|(\Psi_\beta * \varphi_n)(t)|$, $\varphi_n(t) = \text{sgn} \sin nt$. Then*

$$\begin{aligned} d_{2n}(C_{\beta,\infty}^\psi, C) &\geq \|\Psi_\beta * \varphi_n\|_C, \\ d_{2n-1}(L_{\beta,1}^\psi, L) &\geq \|\Psi_\beta * \varphi_n\|_C. \end{aligned}$$

The sufficient conditions for inclusion $\Psi_\beta \in C_{y,2n}$ for kernels of the form (1) were studied in papers [4, 6, 7, 14, 16, 18, 19]. This allowed the authors of mentioned works to apply Theorem 1 and obtain the exact values for widths $d_m(C_{\beta,\infty}^\psi, C)$ and $d_m(L_{\beta,1}^\psi, L)$ in some new cases.

Let's go to the formulation of the main results. For each fixed $q \in (0, 1)$ we denote by n_q the smallest of numbers $n \geq 9$ that satisfy inequality

$$\frac{43}{10(1-q)} q^{\sqrt{n}} + \frac{160}{57(n - \sqrt{n})} \frac{q}{(1-q)^2} \leq \left(\frac{1}{2} + \frac{2q}{(1+q^2)(1-q)} \right) \left(\frac{1-q}{1+q} \right)^{\frac{4}{1-q^2}}. \quad (15)$$

In the above notations the following statements are true.

Theorem 2. *Let $q \in (0, 1)$. Then inequalities (9) and (10) hold for an arbitrary $\beta \in \mathbb{R}$ and for all $n \geq n_q$.*

Proof. According to Theorem 1 it is sufficient to show that for any $q \in (0, 1)$, $\beta \in \mathbb{R}$ and all numbers $n \geq n_q$ the Poisson kernels $P_{q,\beta}(t)$ satisfy the $C_{y,2n}$ condition, where y_0 is the point in which function $|\Phi_{q,\beta,n}(\cdot)|$ ($\Phi_{q,\beta,n}(\cdot) = (P_{q,\beta} * \varphi_n)(\cdot)$ and $\varphi_n(\cdot)$ defined by equality (7)) takes the maximum value, i.e.,

$$|\Phi_{q,\beta,n}(y_0)| = |(P_{q,\beta} * \varphi_n)(y_0)| = \|P_{q,\beta} * \varphi_n\|_C.$$

The function

$$\Phi_{q,\beta,n}(\cdot) = (P_{q,\beta} * \varphi_n)(\cdot) = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} \sin \left((2\nu+1)n \cdot -\frac{\beta\pi}{2} \right),$$

is periodic with period $2\pi/n$ and such that $\Phi_{q,\beta,n}(\cdot + \frac{\pi}{n}) = -\Phi_{q,\beta,n}(\cdot)$. Therefore, the maximum value of π/n -periodic function $|\Phi_{q,\beta,n}(\cdot)|$ on $[0, \frac{\pi}{n})$ reaches at the point $y_0 = \frac{\theta_n\pi}{n}$, where θ_n is the root of the equation (8), $\theta_n \in [0, 1)$. Since $|\Phi_{q,\beta,n}(\cdot)| = |\Phi_{q,\beta+2,n}(\cdot)|$ then, without reducing the generality, we can assume that $\beta \in [0, 2)$. For these values of β the unique on $[0, 1)$ root of equation (8) can be written explicitly as follows

$$\theta_n = 1 - [\beta] - \frac{1}{\pi} \arcsin \frac{(1 - q^{2n}) \cos \frac{\beta\pi}{2}}{\sqrt{1 - 2q^{2n} \cos \beta\pi + q^{4n}}}, \quad \beta \in [0, 2), \quad (16)$$

where $[a]$ is an integer part of number a .

Functions $\Psi_{\beta,1}(t)$ of the form (11) generated by Poisson kernels $\Psi_{\beta}(t) = P_{q,\beta}(t)$ we shall denote by $P_{q,\beta,1}(t)$ and fundamental SK -spline $\overline{S\Psi}_{\beta,1}(y, \cdot)$ — by $\overline{SP}_{q,\beta,1}(y, \cdot)$. We start with obtained in the paper [18] the representation of function $(\overline{S\Psi}_{\beta,1}(y, t))_{\beta}^{\psi}$, according which, when condition $|\lambda_j(y)| \neq 0$, $j = \overline{1, n}$, is true, then for arbitrary $t \in (x_{k-1}, x_k)$ the following equality holds:

$$(\overline{S\Psi}_{\beta,1}(y, t))_{\beta}^{\psi} = \frac{\pi}{4n^2} \left(2 \sum_{j=1}^{n-1} \frac{\sin jt_k \cdot \rho_j(y) - \cos jt_k \cdot \sigma_j(y)}{|\lambda_j(y)|^2 \sin \frac{j\pi}{2n}} + \frac{(-1)^{k+1} \rho_n(y)}{|\lambda_n(y)|^2} \right), \quad (17)$$

where

$$\lambda_j(\cdot) = \frac{1}{n} \sum_{\nu=1}^{2n} e^{ij\nu\pi/n} \Psi_{\beta,1}(\cdot - \frac{\nu\pi}{n}),$$

i is the imaginary unit, $\rho_j(\cdot) = \operatorname{Re}(\lambda_j(\cdot))$, $\sigma_j(\cdot) = \operatorname{Im}(\lambda_j(\cdot))$, $t_k = \frac{k\pi}{n} - \frac{\pi}{2n}$.

Changing the order of summation in the sum on the right-hand side of

equality (17), we have

$$\begin{aligned}
& \sum_{j=1}^{n-1} \frac{\sin jt_k \cdot \rho_j(y) - \cos jt_k \cdot \sigma_j(y)}{|\lambda_j(y)|^2 \sin \frac{j\pi}{2n}} = \\
& = \sum_{j=1}^{n-1} \frac{\sin(n-j)t_k \cdot \rho_{n-j}(y) - \cos(n-j)t_k \cdot \sigma_{n-j}(y)}{|\lambda_{n-j}(y)|^2 \sin \frac{(n-j)\pi}{2n}} = \\
& = (-1)^{k+1} \sum_{j=1}^{n-1} \frac{\cos jt_k \cdot \rho_{n-j}(y) - \sin jt_k \cdot \sigma_{n-j}(y)}{|\lambda_{n-j}(y)|^2 \cos \frac{j\pi}{2n}}. \tag{18}
\end{aligned}$$

Taking into account (17) and (18) for fundamental SK -spline $\overline{S\Psi}_{\beta,1}(y, t) = \overline{SP}_{q,\beta,1}(y, t)$, generated by Poisson kernel $P_{q,\beta}(t)$, where $y = y_0$ we obtain the representation

$$\begin{aligned}
& (\overline{SP}_{q,\beta,1}(y_0, t))_\beta^q = \\
& = \frac{(-1)^{k+1}\pi}{4n^2} \left(2 \sum_{j=1}^{n-1} \frac{\cos jt_k \cdot \rho_{n-j}(y_0) - \sin jt_k \cdot \sigma_{n-j}(y_0)}{|\lambda_{n-j}(y_0)|^2 \cos \frac{j\pi}{2n}} + \frac{\rho_n(y_0)}{|\lambda_n(y_0)|^2} \right), \tag{19}
\end{aligned}$$

where

$$\lambda_l(y_0) = \frac{1}{n} \sum_{\nu=1}^{2n} e^{il\nu\pi/n} P_{q,\beta,1}(y_0 - \frac{\nu\pi}{n}), \quad l = \overline{1, n}. \tag{20}$$

Based on equality (19) write more convenient for further research representation of function $(\overline{SP}_{q,\beta,1}(y_0, t))_\beta^q$. For this aim initially we will show that quantities $\lambda_{n-j}(y_0)$ of the form (20) for $j = \overline{0, n-1}$ can be expressed as follows

$$\lambda_{n-j}(y_0) = e^{-ijy_0} \left(\left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_j(y_0) \right), \tag{21}$$

where

$$r_j(y_0) = \sum_{\nu=1}^3 r_j^{(\nu)}(y_0), \quad (22)$$

$$r_j^{(1)}(y_0) = \frac{q^{3n-j} e^{i(3ny_0 - \frac{(\beta+1)\pi}{2})}}{3n-j} + \sum_{m=2}^{\infty} \left(\frac{q^{(2m+1)n-j} e^{i((2m+1)ny_0 - \frac{(\beta+1)\pi}{2})}}{(2m+1)n-j} + \frac{q^{(2m-1)n+j} e^{-i((2m-1)ny_0 - \frac{(\beta+1)\pi}{2})}}{(2m-1)n+j} \right), \quad (23)$$

$$r_j^{(2)}(y_0) = i \left(\frac{q^{n+j}}{n+j} - \frac{q^{n-j}}{n-j} \right) \cos(ny_0 - \frac{\beta\pi}{2}), \quad (24)$$

$$r_j^{(3)}(y_0) = \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) (|\sin(ny_0 - \frac{\beta\pi}{2})| - 1) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}). \quad (25)$$

Rewrite the kernel $P_{q,\beta,1}$ in the complex form

$$P_{q,\beta,1}(t) = (P_{q,\beta} * B_1)(t) = \sum_{k=1}^{\infty} \frac{q^k}{k} \cos(kt - \frac{(\beta+1)\pi}{2}) = \frac{1}{2} \sum_{k=-\infty}^{\infty}{}' c_k e^{ikt},$$

where

$$c_k = \frac{q^k}{k} e^{-i\frac{(\beta+1)\pi}{2}}, \quad c_{-k} = \frac{q^k}{k} e^{i\frac{(\beta+1)\pi}{2}}, \quad k \in \mathbb{N}, \quad (26)$$

and a prime symbol at sum symbol means the absence of summand with zero index.

Substituting the kernel $P_{q,\beta,1}$ in (20) with it expansion into a complex Fourier series, we get

$$\begin{aligned} \lambda_l(y_0) &= \frac{1}{n} \sum_{\nu=1}^{2n} e^{il\nu\pi/n} \frac{1}{2} \sum_{k=-\infty}^{\infty}{}' c_k e^{ik(y_0 - \nu\pi/n)} = \\ &= \frac{1}{2n} \sum_{\nu=1}^{2n} \sum_{k=-\infty}^{\infty}{}' c_k e^{i(ky_0 + (l-k)\nu\pi/n)} = \\ &= \frac{1}{2n} \sum_{k=-\infty}^{\infty}{}' c_k e^{iky_0} \sum_{\nu=1}^{2n} e^{i((l-k)\nu\pi/n)}. \end{aligned} \quad (27)$$

It is easy to verify that for $k = l - 2mn$, $m \in \mathbb{Z}$, the equality

$$\sum_{\nu=1}^{2n} e^{i((l-k)\nu\pi/n)} = 2n$$

holds. In case $k \neq l - 2mn$, $m \in \mathbb{Z}$, by using the well-known relations

$$\sum_{\nu=1}^{2n} \cos \nu x = \frac{\sin nx \cos(2n+1)x/2}{\sin x/2},$$

$$\sum_{\nu=1}^{2n} \sin \nu x = \frac{\sin nx \sin(2n+1)x/2}{\sin x/2},$$

we obtain

$$\sum_{\nu=1}^{2n} e^{i((l-k)\nu\pi/n)} = 0.$$

Therefore,

$$\sum_{\nu=1}^{2n} e^{i((l-k)\nu\pi/n)} = \begin{cases} 0, & \text{if } k \neq l - 2mn, m \in \mathbb{Z}; \\ 2n, & \text{if } k = l - 2mn, m \in \mathbb{Z}. \end{cases} \quad (28)$$

From (27) and (28) for $l = \overline{1, n}$ the next representation

$$\lambda_l(y_0) = \sum_{m=-\infty}^{+\infty} c_{l-2mn} e^{i(l-2mn)y_0} = \sum_{m=-\infty}^{+\infty} c_{2mn+l} e^{i(2mn+l)y_0}$$

is followed. Hence, for $l = n - j$, $j = \overline{0, n-1}$ we have

$$\begin{aligned} \lambda_{n-j}(y_0) &= \sum_{m=-\infty}^{+\infty} c_{(2m+1)n-j} e^{i((2m+1)n-j)y_0} = \\ &= e^{-ijy_0} (c_{n-j} e^{iny_0} + c_{-(n+j)} e^{-iny_0} + r_j^{(1)}(y_0)). \end{aligned} \quad (29)$$

By using (26), transform first two summands in (29) in the following way:

$$\begin{aligned} c_{n-j} e^{iny_0} + c_{-(n+j)} e^{-iny_0} &= \\ &= \frac{q^{n-j}}{n-j} e^{i(ny_0 - \frac{(\beta+1)\pi}{2})} + \frac{q^{n+j}}{n+j} e^{-i(ny_0 - \frac{(\beta+1)\pi}{2})} = \\ &= \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \cos(ny_0 - \frac{(\beta+1)\pi}{2}) + \\ &+ i \left(\frac{q^{n-j}}{n-j} - \frac{q^{n+j}}{n+j} \right) \sin(ny_0 - \frac{(\beta+1)\pi}{2}) = \\ &= \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \sin(ny_0 - \frac{\beta\pi}{2}) + r_j^{(2)}(y_0). \end{aligned} \quad (30)$$

Writing $\sin(ny_0 - \frac{\beta\pi}{2})$ in the form

$$\sin\left(ny_0 - \frac{\beta\pi}{2}\right) = |\sin(ny_0 - \frac{\beta\pi}{2})| \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}),$$

from (30) we get

$$\begin{aligned} c_{n-j}e^{iny_0} + c_{-(n+j)}e^{-iny_0} &= \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j}\right) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + \\ &+ \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j}\right) (|\sin(ny_0 - \frac{\beta\pi}{2})| - 1) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_j^{(2)}(y_0) = \\ &= \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j}\right) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_j^{(2)}(y_0) + r_j^{(3)}(y_0). \end{aligned} \quad (31)$$

Equalities (29) and (31) yield formula (21).

Taking into account formulas (19) and (21) let establish a new representation for the function $(\overline{SP}_{q,\beta,1}(y_0, t))_\beta^q$.

Lemma 1. *Let $q \in (0, 1)$, $\beta \in \mathbb{R}$, $y_0 = \frac{\theta_n\pi}{n}$, where θ_n is the unique on $[0, 1)$ root of equation (8). Then for arbitrary $t \in (\frac{(k-1)\pi}{n}, \frac{k\pi}{n})$, $k = \overline{1, 2n}$ the following equality holds:*

$$(\overline{SP}_{q,\beta,1}(y_0, t))_\beta^q = (-1)^{k+1} \frac{\pi \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2})}{4nq^n} (\mathcal{P}_q(t_k - y_0) + \sum_{m=1}^5 \gamma_m(y_0)), \quad (32)$$

where $\mathcal{P}_q(t)$ is the Poisson kernel of heat conduction equation

$$\mathcal{P}_q(t) = \frac{1}{2} + 2 \sum_{j=1}^{\infty} \frac{\cos jt}{q^j + q^{-j}}, \quad (33)$$

and

$$\gamma_1(y_0) = \gamma_1(k, y_0) = 2 \sum_{j=[\sqrt{n}]+1}^{n-1} \frac{\cos j(t_k - y_0)}{\frac{n}{q^n} |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}}, \quad (34)$$

$$\gamma_2(y_0) = \gamma_2(k, y_0) =$$

$$= \frac{q^n \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2})}{n} \left(\frac{z_0(y_0)}{|\lambda_n(y_0)|^2} + 2 \sum_{j=1}^{n-1} \frac{z_j(y_0)}{|\lambda_{n-j}(y_0)|^2 \cos \frac{j\pi}{2n}} \right), \quad (35)$$

$$\gamma_3(y_0) = -\frac{R_0(y_0)\frac{n}{q^n}}{2(2 + R_0(y_0)\frac{n}{q^n})}, \quad (36)$$

$$\gamma_4(y_0) = \gamma_4(k, y_0) = -2 \sum_{j=1}^{[\sqrt{n}]} \frac{\delta_j(y_0) \cos j(t_k - y_0)}{\frac{n}{q^n} |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}}, \quad (37)$$

$$\gamma_5(y_0) = \gamma_5(k, y_0) = -2 \sum_{j=[\sqrt{n}]+1}^{\infty} \frac{\cos j(t_k - y_0)}{q^j + q^{-j}}, \quad (38)$$

$$\delta_j(y_0) = \frac{n |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}}{(q^{-j} + q^j)q^n} - 1, \quad j = \overline{0, [\sqrt{n}]}, \quad (39)$$

$$z_j(y_0) = |r_j(y_0)| \cos(j(t_k - y_0) + \arg(r_j(y_0))) - \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) R_j(y_0) \cos(j(t_k - y_0)), \quad j = \overline{0, n-1}, \quad (40)$$

$$R_j(y_0) = |\lambda_{n-j}(y_0)| - \frac{q^{n-j}}{n-j} - \frac{q^{n+j}}{n+j}, \quad j = \overline{0, n-1}, \quad (41)$$

$t_k = \frac{k\pi}{n} - \frac{\pi}{2n}$, quantities $\lambda_l(y_0)$ and $r_j(y_0)$, $j = \overline{0, n-1}$, are defined by equalities (20) and (22) respectively.

Proof. Let transform the numerator of each summand on the right-hand side of equality (19). For this aim by virtue of (21) we write

$$\begin{aligned} \rho_{n-j}(y_0) &= \operatorname{Re}(\lambda_{n-j}(y_0)) = \\ &= \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \cos jy_0 \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + \operatorname{Re}(e^{-ijy_0} r_j(y_0)); \end{aligned} \quad (42)$$

$$\begin{aligned} \sigma_{n-j}(y_0) &= \operatorname{Im}(\lambda_{n-j}(y_0)) = \\ &= - \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \sin jy_0 \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + \operatorname{Im}(e^{-ijy_0} r_j(y_0)). \end{aligned} \quad (43)$$

From equalities (42) and (43) we obtain

$$\begin{aligned} &\cos jt_k \cdot \rho_{n-j}(y_0) - \sin jt_k \cdot \sigma_{n-j}(y_0) = \\ &= \cos jt_k \left(\left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \cos jy_0 \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + \operatorname{Re}(e^{-ijy_0} r_j(y_0)) \right) + \end{aligned}$$

$$\begin{aligned}
& + \sin jt_k \left(\left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \sin jy_0 \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) - \operatorname{Im}(e^{-ijy_0} r_j(y_0)) \right) = \\
& = \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \cos(j(t_k - y_0)) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + \\
& + \cos jt_k \cdot \operatorname{Re}(e^{-ijy_0} r_j(y_0)) - \sin jt_k \cdot \operatorname{Im}(e^{-ijy_0} r_j(y_0)) = \\
& = \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} + R_j(y_0) \right) \cos(j(t_k - y_0)) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + z_j(y_0) = \\
& = |\lambda_{n-j}(y_0)| \cos(j(t_k - y_0)) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + z_j(y_0), \tag{44}
\end{aligned}$$

where

$$\begin{aligned}
z_j(y_0) &= \cos jt_k \cdot \operatorname{Re}(e^{-ijy_0} r_j(y_0)) - \sin jt_k \cdot \operatorname{Im}(e^{-ijy_0} r_j(y_0)) - \\
& - \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) R_j(y_0) \cos(j(t_k - y_0)),
\end{aligned}$$

and $R_j(y_0)$ are defined by (41). By virtue of obvious equality

$$e^{-ijy_0} r_j(y_0) = |r_j(y_0)| (\cos(\arg(r_j(y_0)) - jy_0) + i \sin(\arg(r_j(y_0)) - jy_0))$$

the quantity $z_j(y_0)$ can be represented in form (40).

For $j = 0$ formula (21) transforms into next one:

$$\lambda_n(y_0) = 2 \frac{q^n}{n} \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_0(y_0), \tag{45}$$

where $r_0(y_0)$ is defined by equality (22) in which

$$r_0^{(1)}(y_0) = 2 \sum_{m=2}^{\infty} \frac{q^{(2m-1)n}}{(2m-1)n} \cos((2m-1)ny_0 - \frac{(\beta+1)\pi}{2}), \tag{46}$$

$$r_0^{(2)}(y_0) = 0, \tag{47}$$

$$r_0^{(3)}(y_0) = -2 \frac{q^n}{n} (|\sin(ny_0 - \frac{\beta\pi}{2})| - 1) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}). \tag{48}$$

From (45)–(48) it follows that $\sigma_n(y_0) = 0$ and then

$$\rho_n(y_0) = \lambda_n(y_0) = 2 \frac{q^n}{n} \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_0(y_0).$$

Hence, taking into account (40) and (41), we can write

$$\rho_n(y_0) = \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) \left(2 \frac{q^n}{n} + R_0(y_0) \right) + z_0(y_0) =$$

$$= \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) |\lambda_n(y_0)| + z_0(y_0), \quad (49)$$

where

$$z_0(y_0) = r_0(y_0) - \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) R_0(y_0).$$

As follows from (16) (see also [1, p. 538]), the inclusion $ny_0 \in [\frac{\pi}{2}, \pi)$ is true for $\beta \in [0, 1) \cup [2, 3)$ and the inclusion $ny_0 \in [0, \frac{\pi}{2})$ — for $\beta \in [1, 2) \cup [3, 4)$. Consequently

$$\sin(ny_0 - \frac{\beta\pi}{2}) = \sin ny_0 \cos \frac{\beta\pi}{2} - \cos ny_0 \sin \frac{\beta\pi}{2} \neq 0.$$

From representation (19) and equalities (44) and (49) we have

$$\begin{aligned} & (\overline{SP}_{q,\beta,1}(y_0, t))_\beta^q = \\ &= \frac{(-1)^{k+1}\pi}{4nq^n} \left(\operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) \left(2 \frac{q^n}{n} \sum_{j=1}^{[\sqrt{n}]} \frac{\cos j(t_k - y_0)}{|\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}} + \frac{q^n}{n|\lambda_n(y_0)|} + \right. \right. \\ & \left. \left. + 2 \frac{q^n}{n} \sum_{j=[\sqrt{n}]+1}^{n-1} \frac{\cos j(t_k - y_0)}{|\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}} \right) + 2 \frac{q^n}{n} \sum_{j=1}^{n-1} \frac{z_j(y_0)}{|\lambda_{n-j}(y_0)|^2 \cos \frac{j\pi}{2n}} + \frac{q^n z_0(y_0)}{n|\lambda_n(y_0)|^2} \right) = \\ &= \frac{(-1)^{k+1}\pi}{4nq^n} \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) \times \\ & \times \left(2 \frac{q^n}{n} \sum_{j=1}^{[\sqrt{n}]} \frac{\cos j(t_k - y_0)}{|\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}} + \frac{q^n}{n|\lambda_n(y_0)|} + \gamma_1(y_0) + \gamma_2(y_0) \right). \quad (50) \end{aligned}$$

By virtue of (41)

$$\frac{q^n}{n|\lambda_n(y_0)|} = \frac{1}{2 + R_0(y_0) \frac{n}{q^n}} = \frac{1}{2} - \frac{R_0(y_0) \frac{n}{q^n}}{2(2 + R_0(y_0) \frac{n}{q^n})} = \frac{1}{2} + \gamma_3(y_0). \quad (51)$$

Using formula (39) we can write equalities

$$2 \frac{q^n}{n} \sum_{j=1}^{[\sqrt{n}]} \frac{\cos j(t_k - y_0)}{|\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n}} = 2 \sum_{j=1}^{[\sqrt{n}]} \frac{\cos j(t_k - y_0)}{(q^j + q^{-j})(1 + \delta_j(y_0))} =$$

$$\begin{aligned}
&= 2 \sum_{j=1}^{[\sqrt{n}]} \frac{\cos j(t_k - y_0)}{q^j + q^{-j}} - 2 \sum_{j=1}^{[\sqrt{n}]} \frac{\delta_j(y_0) \cos j(t_k - y_0)}{(q^j + q^{-j})(1 + \delta_j(y_0))} = \\
&= \mathcal{P}_q(t_k - y_0) - \frac{1}{2} + \gamma_4(y_0) + \gamma_5(y_0),
\end{aligned} \tag{52}$$

By combining (50)–(52) we obtain (32). Lemma is proved.

Next lemma contains the lower bound for the minimum value of kernel $\mathcal{P}_q(\cdot)$ of the form (33).

Lemma 2. *Let $q \in (0, 1)$. Then for arbitrary $x \in \mathbb{R}$ the following inequality holds*

$$\mathcal{P}_q(x) > \left(\frac{1}{2} + \frac{2q}{(1 + q^2)(1 - q)} \right) \left(\frac{1 - q}{1 + q} \right)^{\frac{4}{1 - q^2}}. \tag{53}$$

Proof. We use the known in the theory of elliptic functions representation for the kernel $\mathcal{P}_q(x)$ (see, for example, [20, p. 867]):

$$\mathcal{P}_q(x) = \frac{K}{\pi} \operatorname{dn} \frac{Kx}{\pi}, \tag{54}$$

where

$$K = \pi \left(\frac{1}{2} + 2 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{2j}} \right). \tag{55}$$

According to formula (8.146.22) from [20, p. 868], we have

$$\operatorname{dn} \frac{Kx}{\pi} = \exp \left\{ -8 \sum_{j=1}^{\infty} \frac{1}{2j - 1} \frac{q^{2j-1}}{1 - q^{2(2j-1)}} \sin^2 \frac{2j - 1}{2} x \right\}, \tag{56}$$

By virtue of equality

$$\sum_{\nu=1}^{\infty} \frac{q^{2\nu-1}}{2\nu - 1} = \frac{1}{2} \ln \frac{1 + q}{1 - q}$$

(see, for example, [20, p. 53]) the estimate

$$\begin{aligned}
&\sum_{j=1}^{\infty} \frac{1}{2j - 1} \frac{q^{2j-1}}{1 - q^{2(2j-1)}} \sin^2 \frac{2j - 1}{2} x < \\
&< \frac{1}{1 - q^2} \sum_{j=1}^{\infty} \frac{q^{2j-1}}{2j - 1} = \frac{1}{1 - q^2} \frac{1}{2} \ln \frac{1 + q}{1 - q}
\end{aligned} \tag{57}$$

is true.

From (56) and (57) we get the following inequality:

$$\operatorname{dn} \frac{Kx}{\pi} > \left(\frac{1-q}{1+q} \right)^{\frac{4}{1-q^2}}, \quad (58)$$

and from (55) — the inequality

$$\frac{K}{\pi} > \frac{1}{2} + \frac{2}{1+q^2} \sum_{j=1}^{\infty} q^j = \frac{1}{2} + \frac{2q}{(1+q^2)(1-q)}. \quad (59)$$

Formulas (54), (58) and (59) yield (53). Lemma is proved.

The following statement contains the upper bound of the sum $\sum_{k=1}^5 |\gamma_k(y_0)|$.

Lemma 3. *Let $q \in (0, 1)$, $\beta \in \mathbb{R}$ and let the quantities $\gamma_k(y_0)$, $k = \overline{1, 5}$, be defined by equalities (34)–(38). Then if $n \geq 9$ and the condition*

$$\frac{q^n}{1-q^{2n}} \leq \frac{7q^{\sqrt{n}}}{37n^2} \quad (60)$$

is true the following estimate holds

$$\sum_{k=1}^5 |\gamma_k(y_0)| \leq \frac{43}{10(1-q)} q^{\sqrt{n}} + \frac{160}{57(n-\sqrt{n})} \frac{q}{(1-q)^2}.$$

Proof. Let estimate each of summands $|\gamma_k(y_0)|$, $k = \overline{1, 5}$, separately. To do this, first obtain upper bounds of quantities $|r_j(y_0)|$ and $|R_j(y_0)|$ for $j = \overline{0, n-1}$. Taking into account that according to the convexity of sequence $\frac{q^k}{k}$ the inequality $\frac{q^{k-j}}{k-j} + \frac{q^{k+j}}{k+j} < \frac{q^{k-n}}{k-n} + \frac{q^{k+n}}{k+n}$, $j = \overline{0, n-1}$, is true, from (23) we obtain

$$\begin{aligned} |r_j^{(1)}(y_0)| &\leq \frac{q^{3n-j}}{3n-j} + \sum_{m=2}^{\infty} \left(\frac{q^{(2m+1)n-j}}{(2m+1)n-j} + \frac{q^{(2m-1)n+j}}{(2m-1)n+j} \right) = \\ &= \sum_{m=1}^{\infty} \left(\frac{q^{(2m+1)n-j}}{(2m+1)n-j} + \frac{q^{(2m+1)n+j}}{(2m-1)n+j} \right) \leq \\ &\leq \sum_{m=1}^{\infty} \left(\frac{q^{2mn}}{2mn} + \frac{q^{2(m+1)n}}{2(m+1)n} \right) = \end{aligned}$$

$$= \frac{q^{2n}}{2n} + \sum_{m=2}^{\infty} \frac{q^{2mn}}{mn} \leq \frac{1}{2n} \sum_{m=1}^{\infty} q^{2mn} = \frac{q^{2n}}{2n(1-q^{2n})}. \quad (61)$$

Since $y_0 = \frac{\theta_n \pi}{n}$, then from (8) we get

$$\begin{aligned} \left| \cos \left(ny_0 - \frac{\beta \pi}{2} \right) \right| &= \left| \sum_{\nu=1}^{\infty} q^{2\nu n} \cos \left((2\nu+1)ny_0 - \frac{\beta \pi}{2} \right) \right| \leq \\ &\leq \sum_{\nu=1}^{\infty} q^{2\nu n} = \frac{q^{2n}}{1-q^{2n}}. \end{aligned} \quad (62)$$

From (24) and (62) we have

$$|r_j^{(2)}(y_0)| \leq \left| \cos \left(ny_0 - \frac{\beta \pi}{2} \right) \right| \left(\frac{q^{n-j}}{n-j} - \frac{q^{n+j}}{n+j} \right) \leq \frac{q^{2n}}{1-q^{2n}} \left(q - \frac{q^{2n-1}}{2n-1} \right). \quad (63)$$

It follows from (62) that

$$0 \leq 1 - \left| \sin \left(ny_0 - \frac{\beta \pi}{2} \right) \right| \leq \left| \cos \left(ny_0 - \frac{\beta \pi}{2} \right) \right| \leq \frac{q^{2n}}{1-q^{2n}}. \quad (64)$$

By using (25) and (64) we obtain

$$|r_j^{(3)}(y_0)| \leq \frac{q^{2n}}{1-q^{2n}} \left(q + \frac{q^{2n-1}}{2n-1} \right). \quad (65)$$

Therefore, for quantities $r_j(y_0)$ from (61), (63) and (65) follows the estimate

$$|r_j(y_0)| \leq \left| \sum_{\nu=1}^3 r_j^{(\nu)}(y_0) \right| \leq \frac{q^{2n}}{1-q^{2n}} \left(2q + \frac{1}{2n} \right) \leq \frac{37q^{2n}}{18(1-q^{2n})}, j = \overline{0, n-1}. \quad (66)$$

For $j = 0$ we can improve estimate (66). Indeed, by virtue of (46) and (48) when $j = 0$ we have

$$\begin{aligned} |r_0^{(1)}(y_0)| &\leq 2 \sum_{m=2}^{\infty} \frac{q^{(2m-1)n}}{(2m-1)n} \leq \frac{2}{3n} \sum_{m=2}^{\infty} q^{(2m-1)n} = \frac{2}{3n} \frac{q^{3n}}{1-q^{2n}}, \\ |r_0^{(3)}(y_0)| &\leq \frac{2q^{3n}}{n(1-q^{2n})}. \end{aligned}$$

Then, according to (47)

$$|r_0(y_0)| \leq |r_0^{(1)}(y_0) + r_0^{(3)}(y_0)| \leq \frac{8}{3n} \frac{q^{3n}}{1-q^{2n}}. \quad (67)$$

From (21) for the quantity $|R_j(y_0)|$ of form (41) we obtain the following representation:

$$|\lambda_{n-j}(y_0)| = \left| \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} \right) \operatorname{sgn} \sin(ny_0 - \frac{\beta\pi}{2}) + r_j(y_0) \right|,$$

which immediately gives us the estimates

$$|\lambda_{n-j}(y_0)| \leq \frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} + |r_j(y_0)|, \quad (68)$$

$$|\lambda_{n-j}(y_0)| \geq \frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} - |r_j(y_0)|. \quad (69)$$

By virtue of (41), (68) and (69) we obtain

$$|R_j(y_0)| \leq |r_j(y_0)|, \quad j = \overline{0, n-1}. \quad (70)$$

Let's obtain the estimate of the quantity $|\gamma_1(y_0)|$. Taking into account that for $x \in [0, \frac{\pi}{2})$ the inequality $\cos x \geq 1 - \frac{2x}{\pi} > 0$ holds, we get

$$\cos \frac{j\pi}{2n} \geq 1 - \frac{j}{n} = \frac{n-j}{n}, \quad j = \overline{0, n-1}. \quad (71)$$

According to (69), (66) and (71), we have

$$\begin{aligned} \frac{n}{q^n} |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n} &\geq \frac{n}{q^n} \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} - \frac{37q^{2n}}{18(1-q^{2n})} \right) \frac{n-j}{n} = \\ &= q^{-j} + \frac{n-j}{n+j} q^j - \frac{37(n-j)q^n}{18(1-q^{2n})}. \end{aligned} \quad (72)$$

Since for $n \geq 9$ $\frac{9q^{-j}}{10} > \frac{9}{10} > \frac{7}{9} > \frac{7q^{\sqrt{n}}}{n}$, then from condition (60) we obtain the inequality

$$\frac{9q^{-j}}{370n} > \frac{q^n}{1-q^{2n}},$$

which is equivalent to the following inequality:

$$\frac{q^{-j}}{20} > \frac{37nq^n}{18(1-q^{2n})}, \quad j = \overline{0, n-1}. \quad (73)$$

By virtue of (73) the next estimates hold:

$$q^{-j} + \frac{n-j}{n+j} q^j - \frac{37(n-j)q^n}{18(1-q^{2n})} =$$

$$= \frac{19}{20}q^{-j} + \frac{q^{-j}}{20} + \frac{n-j}{n+j}q^j - \frac{37(n-j)q^n}{18(1-q^{2n})} > \frac{19}{20}q^{-j}, \quad j = \overline{0, n-1}. \quad (74)$$

Combining (72) and (74), we conclude that

$$\frac{n}{q^n} |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n} \geq \frac{19q^{-j}}{20}. \quad (75)$$

Therefore, taking into account (75), from (34) we obtain

$$|\gamma_1(y_0)| \leq \frac{40}{19} \sum_{j=[\sqrt{n}]+1}^{n-1} q^j = \frac{40(q^{[\sqrt{n}]+1} - q^{n-1})}{19(1-q)} \leq \frac{40q^{\sqrt{n}}}{19(1-q)}. \quad (76)$$

Let estimate the quantity $|\gamma_2(y_0)|$. Taking into account estimates (66), (69), (71) and (74), we have

$$\begin{aligned} \frac{n}{q^n} |\lambda_{n-j}(y_0)|^2 \cos \frac{j\pi}{2n} &\geq \frac{n}{q^n} \left(\frac{q^{n-j}}{n-j} + \frac{q^{n+j}}{n+j} - \frac{37q^{2n}}{18(1-q^{2n})} \right)^2 \frac{n-j}{n} = \\ &= \frac{q^n}{n-j} \left(q^{-j} + \frac{n-j}{n+j}q^j - \frac{37(n-j)q^n}{18(1-q^{2n})} \right)^2 > \frac{361q^{n-2j}}{400n}. \end{aligned} \quad (77)$$

From (70) and (40) it follows that $|z_j(y_0)| \leq 2|r_j(y_0)|$. Then using (66), (77) and (60), from (35) we conclude

$$\begin{aligned} |\gamma_2(y_0)| &\leq \frac{1600}{361} \max_{0 \leq j \leq n-1} |r_j(y_0)| \frac{n}{q^n} \sum_{j=0}^{n-1} q^{2j} < \frac{29600 n q^n}{3249(1-q^{2n})} \sum_{j=0}^{\infty} q^{2j} \leq \\ &\leq \frac{29600 n}{3249} \frac{7q^{\sqrt{n}}}{37n^2} \frac{1}{1-q^2} \leq \frac{5600}{3249n} q^{\sqrt{n}} \frac{1}{1-q^2}. \end{aligned} \quad (78)$$

Let estimate $|\gamma_3(y_0)|$. When $n \geq 9$ from condition (60) we obtain the inequality

$$q^{2n} \leq \frac{7}{3004}.$$

Then from (36), (60) and (67) we get

$$\begin{aligned} |\gamma_3(y_0)| &\leq \frac{\frac{8q^{2n}}{3(1-q^{2n})}}{2 \left| 2 - \frac{8q^{2n}}{3(1-q^{2n})} \right|} = \frac{2q^{2n}}{3-7q^{2n}} = \frac{1-q^{2n}}{3-7q^{2n}} \frac{2q^{2n}}{1-q^{2n}} = \\ &= \left(\frac{1}{7} + \frac{4}{7(3-7q^{2n})} \right) \frac{2q^{2n}}{1-q^{2n}} < \end{aligned}$$

$$< \frac{3}{7} \frac{2q^{2n}}{1 - q^{2n}} < \frac{6q^{n+\sqrt{n}}}{37n^2}. \quad (79)$$

Before estimating $|\gamma_4(y_0)|$, let's obtain the upper bounds for $|\delta_j(y_0)|$ of form (39). By virtue of (41)

$$\begin{aligned} & \frac{n}{q^n} |\lambda_{n-j}(y_0)| \cos \frac{j\pi}{2n} = \\ & = \left(\frac{n}{n-j} \frac{q^{n-j}}{q^n} + \frac{n}{n+j} \frac{q^{n+j}}{q^n} + R_j(y_0) \frac{n}{q^n} \right) \cos \frac{j\pi}{2n} = \\ & = \left(\left(1 + \frac{j}{n-j}\right) q^{-j} + \left(1 - \frac{j}{n+j}\right) q^j + R_j(y_0) \frac{n}{q^n} \right) \cos \frac{j\pi}{2n} = \\ & = (q^{-j} + q^j) \left(1 - 2 \sin^2 \frac{j\pi}{4n}\right) + \\ & + \left(\frac{j}{n-j} q^{-j} - \frac{j}{n+j} q^j + R_j(y_0) \frac{n}{q^n} \right) \cos \frac{j\pi}{2n}. \end{aligned} \quad (80)$$

From (39), (66), (70), (80) and the convexity of sequence q^k for quantities $|\delta_j(y_0)|$ the next inequalities follows:

$$\begin{aligned} |\delta_j(y_0)| & \leq 2 \sin^2 \frac{j\pi}{4n} + \frac{1}{q^{-j} + q^j} \left(\frac{j}{n-j} q^{-j} + \frac{j}{n+j} q^j + |R_j(y_0)| \frac{n}{q^n} \right) \leq \\ & \leq 2 \left(\frac{j\pi}{4n} \right)^2 + \frac{j}{n-j} + \frac{n|R_j(y_0)|}{q^{n-j} + q^{n+j}} \leq \frac{j^2\pi^2}{8n^2} + \frac{j}{n-j} + \frac{37nq^n}{36(1 - q^{2n})} = \\ & = \frac{4j}{3(n-j)} + \left(\frac{j^2\pi^2}{8n^2} + \frac{37nq^n}{36(1 - q^{2n})} - \frac{j}{3(n-j)} \right). \end{aligned} \quad (81)$$

Prove that expression in parentheses on the right-hand side of (81) is negative, i.e., following inequality holds:

$$\frac{j}{3(n-j)} - \frac{j^2\pi^2}{8n^2} > \frac{37nq^n}{36(1 - q^{2n})}. \quad (82)$$

Then from (81) and (82) we shall have that

$$|\delta_j(y_0)| \leq \frac{4j}{3(n-j)}. \quad (83)$$

Indeed, for any fixed $x \geq 9$ the function $f(x, \tau) = \frac{\tau}{3(x-\tau)} - \frac{\tau^2\pi^2}{8x^2}$ is convex upward on $[1, \sqrt{x}]$. So it takes the smallest value in $\tau = 1$ or $\tau = \sqrt{x}$. Consider

the difference $f(x, 1) - f(x, \sqrt{x})$ for $x \geq 9$:

$$\begin{aligned} f(x, 1) - f(x, \sqrt{x}) &= \frac{1}{3(x-1)} - \frac{\pi^2}{8x^2} - \frac{\sqrt{x}}{3(x-\sqrt{x})} + \frac{x\pi^2}{8x^2} = \\ &= \frac{-8x^2\sqrt{x} + 3\pi^2(x-1)^2}{24x^2(x-1)}. \end{aligned} \quad (84)$$

Since, as it is not hard to verify, the function $g(x) = -8x^2\sqrt{x} + 3\pi^2(x-1)^2$ decreases on $[9, +\infty)$ and $g(9) < 0$ then, by virtue of (84), $f(x, 1) - f(x, \sqrt{x}) < 0$. By using (60), for $n \geq 9$ we have

$$\begin{aligned} \frac{j}{3(n-j)} - \frac{j^2\pi^2}{8n^2} &\geq \frac{1}{3(n-1)} - \frac{\pi^2}{8n^2} > \frac{1}{3n} \left(1 - \frac{\pi^2}{24}\right) > \\ &> \frac{7}{36n} > \frac{7q^{\sqrt{n}}}{36n} > \frac{37nq^n}{36(1-q^{2n})}. \end{aligned}$$

From this (82) is followed as well as (83).

Formulas (37), (75) and (83) allow us to obtain for $n \geq 9$ the following estimate for quantity $\gamma_4(y_0)$:

$$\begin{aligned} |\gamma_4(y_0)| &\leq 2 \sum_{j=1}^{[\sqrt{n}]} \frac{\frac{4j}{3(n-j)}}{\frac{19q^{-j}}{20}} = \frac{160}{57} \sum_{j=1}^{[\sqrt{n}]} \frac{j}{n-j} q^j \leq \\ &\leq \frac{160}{57(n-\sqrt{n})} \sum_{j=1}^{[\sqrt{n}]} jq^j < \frac{160}{57(n-\sqrt{n})} \sum_{j=1}^{\infty} jq^j < \\ &< \frac{160}{57(n-\sqrt{n})} \frac{q}{(1-q)^2}. \end{aligned} \quad (85)$$

By virtue of (38) for quantity $|\gamma_5(y_0)|$ we have

$$|\gamma_5(y_0)| \leq 2 \sum_{j=[\sqrt{n}]+1}^{\infty} q^j = 2 \frac{q^{[\sqrt{n}]+1}}{1-q} < 2 \frac{q^{\sqrt{n}}}{1-q}. \quad (86)$$

Taking into account estimates (76), (78), (79), (85) and (86), for $n \geq 9$ we obtain

$$\sum_{k=1}^5 |\gamma_k(y_0)| <$$

$$\begin{aligned}
&< \frac{40q^{\sqrt{n}}}{19(1-q)} + \frac{5600}{3249n} q^{\sqrt{n}} \frac{1}{1-q^2} + \frac{6q^{n+\sqrt{n}}}{37n^2} + \frac{160}{57(n-\sqrt{n})} \frac{q}{(1-q)^2} + \frac{2q^{\sqrt{n}}}{1-q} < \\
&< \frac{q^{\sqrt{n}}}{1-q} (2,1053 + 0,1916 + 0,0021 + 2) + \frac{160}{57(n-\sqrt{n})} \frac{q}{(1-q)^2} < \\
&< \frac{43}{10(1-q)} q^{\sqrt{n}} + \frac{160}{57(n-\sqrt{n})} \frac{q}{(1-q)^2}.
\end{aligned}$$

Lemma is proved.

From proven above lemmas 2–3 it follows that for $n \geq 9$ when conditions (15) and (60) are true, we have

$$\mathcal{P}_q(t_k - y_0) + \sum_{m=1}^5 \gamma_m(y_0) \geq 0. \quad (87)$$

By virtue of representation (32) and inequality (87) we conclude that for $n \geq 9$ when conditions (15) and (60) are true the inclusion $P_{q,\beta}(t) \in C_{y_0,2n}$ holds.

Notice that for $q \in (0, \frac{3}{10}]$ condition (60) is true for arbitrary $n \geq 9$. To verify this it is enough to show that sequence $\xi(n) = (n - \sqrt{n}) \ln \frac{3}{10} + 2 \ln n - \ln \left(1 - \left(\frac{3}{10} \right)^{18} \right) \frac{7}{37}$ is monotonically decreasing for $n \geq 9$ and $\xi(9) < 0$. That's why for $n \geq 9$

$$(n - \sqrt{n}) \ln \frac{3}{10} + 2 \ln n - \ln \left(1 - \left(\frac{3}{10} \right)^{18} \right) \frac{7}{37} < 0. \quad (88)$$

Inequality (88) is an equivalent to the next one

$$\frac{\left(\frac{3}{10} \right)^{n-\sqrt{n}}}{1 - \left(\frac{3}{10} \right)^{18}} < \frac{7}{37n^2},$$

and thus for $q \in (0, \frac{3}{10}]$ we get

$$\frac{q^{n-\sqrt{n}}}{1 - q^{2n}} < \frac{\left(\frac{3}{10} \right)^{n-\sqrt{n}}}{1 - \left(\frac{3}{10} \right)^{18}} < \frac{7}{37n^2}.$$

Therefore, for completion the proof of the theorem it remains to show that for $n \geq 9$ and $q \in (\frac{3}{10}, 1)$ the following implication holds:

$$(15) \Rightarrow (60). \quad (89)$$

Since

$$\frac{1}{2} + \frac{2q}{(1+q^2)(1-q)} \geq \frac{1+q}{2(1-q)},$$

then the inequality

$$\frac{160}{57(n - \sqrt{n})} \frac{q}{(1-q)^2} < \frac{1}{2} \left(\frac{1-q}{1+q} \right)^{\frac{4}{1-q^2}-1},$$

follows from (15) and thus,

$$n - \sqrt{n} - \frac{320q}{57(1-q)^2} \left(\frac{1+q}{1-q} \right)^{\frac{4}{1-q^2}-1} > 0. \quad (90)$$

For $n \in \mathbb{N}$ the inequality (90) is true iff

$$\sqrt{n} > \frac{1 + \sqrt{1 + \frac{1280q}{57(1-q)^2} \left(\frac{1+q}{1-q} \right)^{\frac{4}{1-q^2}-1}}}{2}. \quad (91)$$

From (91) it follows that

$$n > \frac{320q}{57(1-q)^2} \left(\frac{1+q}{1-q} \right)^3. \quad (92)$$

Therefore, for $n \geq 9$ and $q \in (0, 1)$

$$(15) \Rightarrow (92). \quad (93)$$

Further show that for $n \geq 9$ and $q \in (0, 1)$ the inequality (60) follows from

$$n > \left(\frac{9(1+q)}{4(1-q)} \right)^2. \quad (94)$$

Since (see, for example, [20, p. 53]) for an arbitrary $q \in (0, 1)$

$$\ln \frac{1}{q} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{1-q}{1+q} \right)^{2k-1} > 2 \frac{1-q}{1+q},$$

then

$$\left(\frac{9(1+q)}{4(1-q)}\right)^2 > \left(\frac{9}{4\frac{1-q}{1+q}}\right)^{\frac{125}{79}} > \left(\frac{9}{2\ln 1/q}\right)^{\frac{125}{79}}. \quad (95)$$

From (94) and (95) the inequality

$$n > \left(\frac{9}{2\ln 1/q}\right)^{\frac{125}{79}},$$

is followed which is equivalent to next one

$$\frac{2}{3}n \ln \frac{1}{q} > 3n^{\frac{46}{125}}. \quad (96)$$

Since for $n \in \mathbb{N}$ $\ln n < n^{\frac{46}{125}}$ and for $n \geq 9$ $1 - \frac{1}{\sqrt{n}} \geq \frac{2}{3}$, then from (96) it follows that

$$n \left(1 - \frac{1}{\sqrt{n}}\right) \ln \frac{1}{q} > 3 \ln n. \quad (97)$$

For $n \geq 9$ from (97) we get

$$\frac{1}{q^n} > \frac{n^3}{q\sqrt{n}} > \frac{9n^2}{q\sqrt{n}} > \frac{38n^2}{7q\sqrt{n}} = \frac{37n^2}{7q\sqrt{n}} + \frac{n^2}{7q\sqrt{n}} > \frac{37n^2}{7q\sqrt{n}} + q^n.$$

Therefore, for $n \geq 9$ and $q \in (0, 1)$

$$(94) \Rightarrow (60). \quad (98)$$

It remains to prove that for $q \in (\frac{3}{10}, 1)$ and $n \geq 9$ holds

$$(92) \Rightarrow (94). \quad (99)$$

For this aim consider the difference of right-hand sides in inequalities (92) and (94), denoting

$$\begin{aligned} v(q) &= \frac{320q}{57(1-q)^2} \left(\frac{1+q}{1-q}\right)^3 - \left(\frac{9(1+q)}{4(1-q)}\right)^2 = \\ &= \left(\frac{1+q}{1-q}\right)^2 \left(\frac{320q(1+q)}{57(1-q)^3} - \left(\frac{9}{4}\right)^2\right). \end{aligned} \quad (100)$$

Since $q \in (\frac{3}{10}, 1)$, then

$$\frac{320q(1+q)}{57(1-q)^3} - \left(\frac{9}{4}\right)^2 > 0. \quad (101)$$

From (100) and (101) we obtain inequality $v(q) > 0$ as well as (99). For $q \in (\frac{3}{10}, 1)$ (89) follows from (93), (98) and (99). Theorem is proved.

Theorem 3. *Let $q \in (0, 1)$. Then for an arbitrary $\beta \in \mathbb{R}$ and all numbers $n \geq n_q$ the following equalities holds:*

$$\begin{aligned} d_{2n}(C_{\beta,\infty}^q, C) &= d_{2n-1}(C_{\beta,\infty}^q, C) = d_{2n-1}(C_{\beta,1}^q, L) = E_n(C_{\beta,\infty}^q)_C = E_n(C_{\beta,1}^q)_L = \\ &= \|P_{q,\beta} * \varphi_n\|_C = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} \sin \left((2\nu+1)\theta_n\pi - \frac{\beta\pi}{2} \right) \right|, \end{aligned} \quad (102)$$

where $\theta_n = \theta_n(q, \beta)$ is the unique on $[0, 1)$ root of equation (8).

In particular, for $n \geq n_q$ and $\beta \in \mathbb{Z}$ the next equalities are true:

$$\begin{aligned} d_{2n}(C_{\beta,\infty}^q, C) &= d_{2n-1}(C_{\beta,\infty}^q, C) = d_{2n-1}(C_{\beta,1}^q, L) = E_n(C_{\beta,\infty}^q)_C = \\ &= E_n(C_{\beta,1}^q)_L = \frac{4}{\pi} \operatorname{arctg} q^n, \quad \beta = 2k, \quad k \in \mathbb{Z}; \end{aligned} \quad (103)$$

$$\begin{aligned} d_{2n}(C_{\beta,\infty}^q, C) &= d_{2n-1}(C_{\beta,\infty}^q, C) = d_{2n-1}(C_{\beta,1}^q, L) = E_n(C_{\beta,\infty}^q)_C = \\ &= E_n(C_{\beta,1}^q)_L = \frac{2}{\pi} \ln \frac{1+q^n}{1-q^n}, \quad \beta = 2k-1, \quad k \in \mathbb{Z}. \end{aligned} \quad (104)$$

Proof. According to Theorem 2 for $n \geq n_q$ inequalities (9) and (10) hold, which in combination with formulas (6), (5) and relation $d_{2n-1}(C_{\beta,\infty}^q, C) \geq d_{2n}(C_{\beta,\infty}^q, C)$ for $\beta \in \mathbb{R}$ yield (102). Let prove (103) and (104).

As follows from (16), for $\beta = 2k$, $k \in \mathbb{Z}$, value $\theta_n = \frac{1}{2}$ is the unique on $[0, 1)$ root of equation (8). Since in this case

$$\begin{aligned} &\sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} \sin \left((2\nu+1)\theta_n\pi - \frac{\beta\pi}{2} \right) = \\ &= (-1)^k \sum_{\nu=0}^{\infty} (-1)^\nu \frac{q^{(2\nu+1)n}}{2\nu+1} = (-1)^k \operatorname{arctg} q^n, \end{aligned}$$

then from (102) we get (103).

For $\beta = 2k-1$, $k \in \mathbb{Z}$, the root of the equation (8), as follows from (16), is value $\theta_n = 0$. Since

$$\sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} \sin \left((2\nu+1)\theta_n\pi - \frac{\beta\pi}{2} \right) =$$

$$= (-1)^k \sum_{\nu=0}^{\infty} \frac{q^{(2\nu+1)n}}{2\nu+1} = (-1)^k \frac{1}{2} \ln \frac{1+q^n}{1-q^n},$$

them from (102) we obtain equalities (104). Theorem is proved.

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